Theorem (Generalized Cauchy Integral Theorem)

Suppose that f is analytic interior to and on a simple closed positively oriented contour C. If z_0 is interior to C, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz.$$

Proof. By induction. The base case ($n=0$) is just Cauchy's Integral formula. Let $n \geq 0$ and assume that the formula holds for n. We need to prove that

$$f^{(n+1)}(z_0) \stackrel{\text{def}}{=} \lim_{\Delta z \to 0} \frac{f^{(n)}(z_0+\Delta z) - f^{(n)}(z_0)}{\Delta z}$$

$$= \frac{(n+1)!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{(n+1)+1}}\, dz.$$

Assume $|\Delta z|$ is so small that $z_0 + \Delta z$ is interior to C. Then by the inductive hypothesis

$$f^{(n)}(z_0+\Delta z) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0-\Delta z)^{n+1}}\, dz$$

and

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz.$$

Use the identity ($A, B \in \mathbb{C}$),

$$A^{n+1} - B^{n+1} = (A-B)(A^n + A^{n-1}B + \cdots + AB^{n-1} + B^n)$$

with $A = \frac{1}{z-z_0-\Delta z}$ and $B = \frac{1}{z-z_0}$. Then

$$\lim_{\Delta z \to 0} \frac{f^{(n)}(z_0+\Delta z) - f^{(n)}(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{n!}{2\pi i} \int_C \frac{f(z)}{\Delta z}\left(\frac{1}{(z-\Delta z - z_0)^{n+1}} - \frac{1}{(z-z_0)^{n+1}}\right) dz$$

$$= \lim_{\Delta z \to 0} \frac{n!}{2\pi i} \int_C \frac{f(z)}{\Delta z}\left(\frac{1}{z-z_0-\Delta z} - \frac{1}{z-z_0}\right)\left(A^n + A^{n-1}B + \cdots + B^n\right) dz$$

$$= \lim_{\Delta z \to 0} \frac{n!}{2\pi i} \int_C \frac{f(z)}{\Delta z}\left(\frac{z-z_0-(z-z_0-\Delta z)}{(z-z_0-\Delta z)(z-z_0)}\right)\left(\quad\text{———}\quad\right) dz$$

$$= \lim_{\Delta z \to 0} \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)}\left(A^n + A^{n-1}B + \cdots + B^n\right) dz$$

$$= \frac{n!}{2\pi i} \int_C \lim_{\Delta z \to 0} \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)}\left(A^n + A^{n-1}B + \cdots + B^n\right) dz$$

$$= \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^2}\, \frac{(n+1)}{(z-z_0)^n}\, dz$$

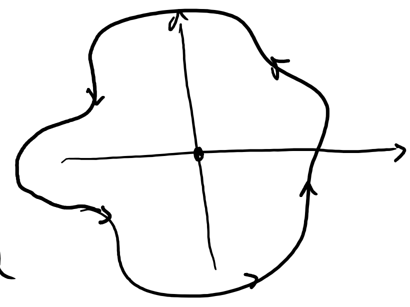
$$= \frac{(n+1)!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{(n+1)+1}}\, dz.$$

This completes the proof. $\blacksquare$

$\boxed{\text{Example (C.f. PSet 4 P5)}}$ Compute the integral

$$\frac{1}{2\pi i} \quad \frac{(1+z)^n}{z^{k+1}}\, dz$$

where C is any simple closed positively oriented contour whose interior contains 0 and $0 \le k \le n$.



Let $f(z) = (1+z)^n$. Since f is entire, f is analytic inside and interior to C. Since 0 is interior to C, the generalized

Cauchy Integral formula applies:

$$\frac{1}{2\pi i}\int_C \frac{(1+z)^n}{z^{k+1}}\,dz = \frac{1}{k!}\left(\frac{k!}{2\pi i}\int_C \frac{(1+z)^n}{z^{k+1}}\,dz\right)$$
$$= \frac{1}{k!} f^{(k)}(0).$$

We have $f^{(k)}(z) = n(n-1)(n-2)\cdots(n-k+1)(1+z)^{n-k}$

$$\Rightarrow f^{(k)}(0) = n(n-1)(n-2)\cdots(n-k+1)$$
$$= \frac{n!}{(n-k)!}$$

Hence, $\frac{1}{2\pi i}\int_C \frac{(1+z)^n}{z^{k+1}}\,d = \frac{1}{k!}\frac{n!}{(n-k)!} = \binom{n}{k}.$

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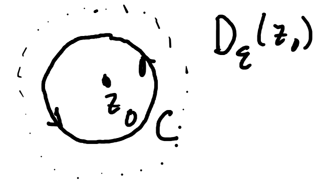
$\boxed{\text{Theorem (Derivatives of Analytic Functions are Analytic)}}$

Suppose that f is analytic at $z_0 \in \mathbb{C}$. Then for all $n \in \mathbb{N}$, $f^{(n)}$ is analytic at z_0.

Proof. Suppose f is analytic $z_0 \in \mathbb{C}$. Choose an open disk $D_q(z_0)$ on which f is analytic. We need to show that there is a neighborhood of z_0 where $f''(z)$ exists for all z in that neighborhood. Let C be the positively oriented circle of radius $\frac{q}{2}$ centered at z_0. Then f is analytic inside and interior to C, so by the generalized Cauchy Integral theorem,



$$f''(z) = \frac{2!}{2\pi i}\int_C \frac{f(s)}{(s-z)^3}\,ds$$

for any z interior to C. Thus, f' has a derivative everywhere in the open set $D_{\varepsilon/2}(z_0)$. Thus, f' is analytic at z_0. By induction, $f^{(n)}$ is analytic for all $n \in \mathbb{N}$.

■

$\boxed{\text{Corollary}}$ If $f(z) = u(x,y) + iv(x,y)$ is analytic at $z = x+iy$, then u and v have continuous partial derivatives of all orders at (x,y).

$\boxed{\text{Theorem (Morera's Theorem)}}$ Suppose f is continuous on a domain D. If

$$\int_C f(z)\,dz = 0$$

for every closed contour in D, then f is analytic on D.

Proof. By the Fundamental theorem of Contour Integrals, there exists an analytic function $F : D \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in D$. But F' is analytic on D by the preceding theorem. So f is analytic on D.

□

When D is simply connected, Morera's theorem is just the converse of the Cauchy-Goursat theorem for simply connected domains

$\boxed{\text{Theorem (Cauchy's Inequalities)}}$ Suppose that f is analytic interior to and on a positively oriented circle $C_R(z_0)$. Then

$$|f^{(n)}(z_0)| \leq \frac{n!}{R^n} \max_{z \in C_R(z_0)} |f(z)|.$$

Proof. By the generalized Cauchy Integral formula:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_R(z_0)} \frac{f(z)}{(z-z_0)^{n+1}}\, dz.$$

Hence,

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi} \left| \int_{C_R(z_0)} \frac{f(z)}{(z-z_0)^{n+1}}\, dz \right|$$

Triangle Ineq

$$\leq \frac{n!}{2\pi} \max_{z \in C_R(z_0)} \frac{|f(z)|}{|z-z_0|^{n+1}} \cdot 2\pi R$$

$$= \frac{n!}{2\pi} \cdot \max_{z \in C_R} |f(z)| \cdot \frac{2\pi R}{R^{n+1}}$$

$$= \frac{n!}{R^n} \max_{z \in C_R} |f(z)|.$$

■

Liouville's Theorem and Fundamental Theorem of Algebra

As an application, we will prove that every nonconstant polynomial with complex coefficients has a root in $\mathbb{C}$. In the language of algebra, this just means that $\mathbb{C}$ is algebraically closed. Thus, the theorem is "purely algebraic", although there is no "purely algebraic" proof.

The proof relies on the following theorem:

$\boxed{\text{Theorem (Liouville's Theorem)}}$ Every bounded entire function is constant.

Proof. The strategy is to show that $f'(z) = 0$ for all $z \in \mathbb{C}$. This is sufficient to prove that f is constant on $\mathbb{C}$, since $\mathbb{C}$ is a domain. Let $z_0 \in \mathbb{C}$.

Since f is bounded, choose $M > 0$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Let $C_R(z_0)$ be a circle of radius $R > 0$ centered at z_0. Then f is analytic inside and on $C_R(z_0)$, so by the Cauchy Inequality,

$$|f'(z_0)| \leq \frac{1!}{R} \max_{z \in C_R(z_0)} |f(z)|$$

$$\leq \frac{M}{R} \xrightarrow{\text{as } R \to \infty} 0$$

Hence $|f'(z_0)| = 0 \Rightarrow f'(z_0) = 0$. This proves the claim. ∎

$\boxed{\text{Theorem (Fundamental Theorem of Algebra)}}$ Any polynomial

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n, \quad a_n \neq 0, \quad a_1, \ldots, a_n \in \mathbb{C}$$

with degree $n \geq 1$ has at least one root in $\mathbb{C}$.

Proof. Suppose to contrary that $p(z)$ has no root in $\mathbb{C}$. This means that $p(z) \neq 0$ for all $z \in \mathbb{C}$. Hence $\dfrac{1}{p(z)}$ is entire. We show that $\frac{1}{p(z)}$ is bounded. By the lemma from week 1, choose $R>0$ such that

$$\left|\frac{1}{p(z)}\right| < \frac{2}{|a_n| R^n} \quad \text{for all } |z|>R.$$

So $\frac{1}{p(z)}$ is bounded outside of the closed disk $\overline{D_R(0)}$. But $\overline{D_R(0)}$ is compact (closed and bounded) and $\frac{1}{p(z)}$ is continuous on $\overline{D_R(0)}$, so $\frac{1}{p(z)}$ is bounded $\overline{D_R(0)}$ by the extreme value theorem. Hence, $\frac{1}{p(z)}$ is bounded on $\mathbb{C}$. By Liouville's theorem, $\frac{1}{p(z)}$ is constant, say

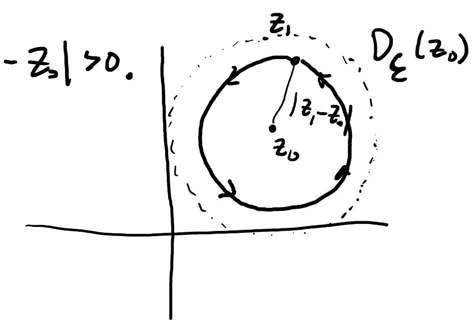
$$\frac{1}{p(z)} = c, \quad \text{for some } c \in \mathbb{C}.$$

Then $p(z) = \frac{1}{c}$, a constant! This contradicts our assumption. ∎

Maximum Modulus Principle

Lemma Suppose that $|f(z)| \leq |f(z_0)|$ at each point z in an analytic neighborhood $D_\varepsilon(z_0)$ of f. Then $f(z) = f(z_0)$ on $D_\varepsilon(z_0)$.

Proof. Let $z_1 \in D_\varepsilon(z_0) \setminus z_0$. Then set $\rho = |z_1 - z_0| > 0$. Let $C_\rho(z_0)$ be the circle of radius $\rho > 0$ centered at z_0. By the Cauchy Integral



$$|f(z_0)| = \left| \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)}{z-z_0}\, dz \right|$$

$$= \frac{1}{2\pi} \left| \int_{C_\rho} \frac{f(z)}{z-z_0}\, dz \right| \qquad z(t) = z_0 + \rho e^{it}$$

$$t \in [0, 2\pi]$$

$$= \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{f(z_0 + \rho e^{it})}{\rho e^{it}} \cdot \rho i e^{it}\, dt \right|$$

$$= \frac{1}{2\pi} \left| \int_0^{2\pi} f(z_0 + \rho e^{it})\, dt \right|$$

$$\underset{\text{T.I.}}{\leq} \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{it})|\, dt$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)|\, dt = |f(z_0)|$$

This proves

$$\frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{it})|\, dt = |f(z_0)|.$$

Rewrite this

$$\int_0^{2\pi} |f(z_0 + \rho e^{it})| - |f(z_0)|\, dt = 0$$

Notice that $|f(z_0 + \rho e^{it})| - |f(z_0)| \geq 0$ for $t \in [0, 2\pi]$. The integrand is also continuous in t. Thus, we must have

$$|f(z_0 + \rho e^{it})| = |f(z_0)| \quad \text{on } [0, 2\pi].$$

Hence, $f(z) = f(z_0)$ for all $z \in C_\rho(z_0)$. By varying the radius ρ, we obtain $f(z) = f(z_0)$ for all $z \in D_\varepsilon(z_0)$. ∎